COMPARISON OF THE BERGMAN AND SZEGÖ KERNELS

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Abstract. The quotient of the Szegö and Bergman kernels for a smooth bounded pseudoconvex domains in $\mathbb{C}^n$ is bounded from above by $\delta|\log\delta|^p$ for any $p > n$, where δ is the distance to the boundary. For a class of domains that includes those of D'Angelo finite type and those with plurisubharmonic defining functions, the quotient is also bounded from below by $\delta|\log\delta|^p$ for any $p < -1$. Moreover, for convex domains, the quotient is bounded from above and below by constant multiples of δ.

Mathematics Subject Classification (2000): 32A25, 32W05, 32U35.
Keywords: Bergman kernel, Szegö kernel, $\bar\partial$-operator, L^2-estimate, Diederich-Fornæss exponent, pluricomplex Green function.

1. Introduction

The Bergman and Szegö kernels are two important reproducing kernels in complex analysis. They are related yet distinct. Whereas the Bergman kernel K (as a measure) is biholomorphically invariant, the Szegö kernel S is not. The former is connected to the $\overline{\partial}$-problem and the $\overline{\partial}$-Neumann Laplacian and the latter the $\bar\partial_b$-problem and the Kohn Laplacian. In his book published in 1972, Stein posted the following problem: What are the relations between K and S? He further noted that the relation between K and S was known only in very special circumstances ([42, p. 20]).

There has been an extensive literature that connects the $\overline{\partial}$-Neumann Laplacian to boundary pseudo-differential operators associated with the Kohn Laplacian (cf. [22, 34, 29]) and mapping properties of the Szegö projection to that of the Bergman projection (cf. [5, 6, 30, 38]). However, there are few results, as far as we know, that directly relate these two kernels.

In this paper, we study boundary behavior of the quotient $S(z,z)/K(z,z)$ of the Szegö and Bergman kernels for a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$. When Ω is strictly pseudoconvex, boundary limiting behavior of the Bergman kernel $K(z,z)$ was obtained by Hörmander [26] and asymptotic expansions for the Bergman and Szegö kernels were established by Fefferman [20] and Boutet-Sjöstrand [7]. As a result, $S(z,z)/K(z,z)$ is asymptotically $\delta(z)/n$ near the boundary, where $\delta(z)$ is the Euclidean distance to the boundary. When Ω is a pseudoconvex domain of finite type in $\mathbb{C}^2$ or a convex domain of finite type in $\mathbb{C}^n$, estimates of the Bergman kernel on diagonal from above and below were obtained by Catlin [11] and J. Chen [13]. Estimates for the Bergman and Szegö kernels (on and off diagonal) and their derivatives from above were established by McNeal[31, 32], Nagel

Research supported in part by NSF grant DMS-0805852, a U.S.-China Collaboration in Mathematical Research supplementary to NSF grant DMS-0500909, and by Fok Ying Tung Education Foundation Grant 111004.

et al [34], and McNeal-Stein [33]. It follows that on these domains, $S(z,z)/K(z,z) \le C\delta(z)$ for some positive constant C.

Our main result can be stated as follows:

Theorem 1.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-smooth boundary.*

(1) *For any $a \in (0,\ 1)$, there exists a constant $C > 0$ such that*
$$\frac{S(z,z)}{K(z,z)} \le C\delta(z)|\log(\delta(z))|^{n/a}.$$

(2) *If there exist a neighborhood U of $b\Omega$, a bounded continuous plurisubharmonic function φ on $U \cap \Omega$, and a defining function ρ of Ω satisfying $i\partial\overline{\partial}\varphi \ge i\rho^{-1}\partial\overline{\partial}\rho$ on $U \cap \Omega$ as currents, then there exist constants $a \in (0,\ 1]$ and $C > 0$ such that*
$$\frac{S(z,z)}{K(z,z)} \ge C\delta(z)|\log(\delta(z))|^{-1/a}.$$

The constant a in the second part of the theorem is a Diederich-Fornæss exponent ([16]): Namely, there exists a negative plurisubharmonic function φ on Ω such that $C_1\delta^a(z) \le -\varphi(z) \le C_2\delta^a(z)$ for some positive constants C_1 and C_2. It was shown by Catlin [9, 10] that any smooth bounded pseudoconvex domain of D'Angelo finite type satisfies Property (P). Sibony further showed that for a smooth bounded pseudoconvex domain satisfying Property (P), the Diederich-Fornæss index, the supremum of the Diederich-Fornæss exponents, is one (see [40, Theorem 2.4]). More recently, Fornæss and Herbig [21] showed that a smooth bounded domain with a defining function that is plurisubharmonic on the boundary also has Diederich-Fornæss index one.

For the convenience of the discussion, a bounded domain that satisfies the condition in (2) will be called *δ-regular*. As we will show in Section 5, such a domain is necessarily hyperconvex with a positive Diederich-Fornæss index. It is easy to see that the class of δ-regular domains includes smooth bounded pseudoconvex domains with a defining function that is plurisubharmonic on $b\Omega$, and it is a consequence of the above-mentioned work of Catlin [10] that this class of domains also includes pseudoconvex domains of D'Angelo finite type (see Proposition 5.2 below). Therefore, in light of these and Theorem 1.1, we have:

Theorem 1.2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose that $b\Omega$ is either of D'Angelo finite type or has a defining function that is plurisubharmonic on $b\Omega$. Then for any constant $a \in (0,\ 1)$, there exist positive constants C_1 and C_2 such that*

$$C_1\delta(z)|\log(\delta(z))|^{-1/a} \le \frac{S(z,z)}{K(z,z)} \le C_2\delta(z)|\log(\delta(z))|^{n/a}. \tag{1.1}$$

The logarithmic terms in the above theorems do not materialize when the domain is convex. More precisely, we have:

Theorem 1.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded convex domain with C^2-smooth boundary. Then there exist positive constants C_1 and C_2 such that*

$$C_1\delta(z) \le S(z,z)/K(z,z) \le C_2\delta(z). \tag{1.2}$$

Our analysis depends on the L^2-estimates for the $\bar{\partial}$-operator by Hörmander [26], Demailly [14], and Berndtsson [2]. We also make essential use of Blocki's estimates for the pluricomplex Green function on hyperconvex domains [4].

This paper is organized as follows: In Section 2, we establish necessary background for the Hardy spaces, the Bergman and Szegö kernels. In Section 3, we review the relevant

L^2-estimates of the $\bar{\partial}$-operator by Hörmander [26], Demailly [14], and Berndtsson [2]. The first part of Theorem 1.1 is proved in Section 4 and the second part in Section 5.

2. Preliminaries

We first establish necessary harmonic analysis background. We refer the reader to [42, 43] for an extensive treatise on the subject. Let D be a bounded domain in $\mathbb{R}^N$ with C^2-smooth boundary. Let $D_\varepsilon = \{x \in D \mid \delta_D(x) > \varepsilon\}$, where $\delta_D(x)$ denotes the Euclidean distance to the boundary bD. For $1 < p < \infty$, the harmonic Hardy space $h^p(D)$ is the space of harmonic functions f such that

$$\|f\|_{h^p}^p = \limsup_{\varepsilon \to 0^+} \int_{bD_\varepsilon} |f(z)|^p \, dS < \infty.$$

The level sets bD_ε in the above definition can be replaced by those of any defining function of D (see [42]). A classical result says that the non-tangential limit $f^*(y)$ of f exists for almost every point y on bD. Furthermore, $f^* \in L^p(bD)$, $\|f\|_{h^p} = \|f^*\|_{L^p(bD)}$, and

$$f(x) = \int_{bD} P(x,y) f^*(y) \, dS(y),$$

where $P(x,y)$ is the Poisson kernel of D.

Throughout the paper, we will use C, together with subscripts, to denote positive constants which could be different in different appearances. We will need the following two simple lemmas.

Lemma 2.1. *Let $D_1 \subset D_2$ be bounded domains in $\mathbb{R}^N$ with C^2-smooth boundaries. There exists a positive constant C such that*

$$\|f\|_{h^p(D_1)} \le C \|f\|_{h^p(D_2)} \tag{2.1}$$

for any $f \in h^p(D_2)$.

Proof. The proof follows the same lines of arguments as in the proof of Theorem 1 in [42]. We provide the detail below. Let

$$g(x) = \int_{bD_2} P_2(x,y) |f^*(y)|^p \, dS(y),$$

where $P_2(x,y)$ is the Poisson kernel of D_2. Then for any $x \in D_2$,

$$|f(x)|^p = \left| \int_{bD_2} P_2(x,y) f^*(y) \, dS(y) \right|^p \le \int_{bD_2} P_2(x,y) |f^*(y)|^p \, dS(y) = g(x).$$

Thus $g(x)$ is a harmonic majorant of $|f|^p$ on D_2. Now fix a point x_0 in D_1. Let $G_1(x,y)$ be the Green function of D_1. Let $D_1^\varepsilon = \{x \in D_1 \mid G_1(x,y) > \varepsilon\}$. Then $-\partial G_1(x,y)/\partial \nu_y = P_\varepsilon(x,y)$ is the Poisson kernel of D_1^ε, where ν_y is the outward normal direction on bD_1^ε. Let $\pi_\varepsilon \colon bD_1^\varepsilon \to bD_1$ be the projection along the normal direction. Since $P_\varepsilon(x_0, \pi_\varepsilon^{-1}(y))$ converges uniformly on bD_1 to $P_1(x_0,y)$ and $C_1 = \min\{P_1(x_0,y) \mid y \in bD_1\} > 0$, we have

$$g(x_0) = \int_{bD_1^\varepsilon} P_\varepsilon(x_0,y) g(y) \, dS \ge \frac{C_1}{2} \int_{bD_1^\varepsilon} g(y) \, dS$$

for sufficiently small $\varepsilon > 0$. It follows that

$$\int_{bD_1^\varepsilon} |f(x)|^p \, dS \le \int_{bD_1^\varepsilon} g(x)\, dS \le \frac{2}{C_1} g(x_0) = \frac{2}{C_1} \int_{bD_2} P_2(x_0, y) |f^*(y)|^p \, dS$$
$$\le \frac{2C_2}{C_1} \int_{bD_2} |f^*(y)|^p \, dS = \frac{2C_2}{C_1} \|f\|^p_{h^p(D_2)},$$

where $C_2 = \max\{P_2(x_0, y) \mid y \in bD_2\} < \infty$. Thus (2.1) holds with $C = (2C_2/C_1)^{1/p}$. □

In what follows, we will also use f to denote the boundary values f^* for $f \in h^p(D)$.

Lemma 2.2. *Let D be a bounded domain in $\mathbb{R}^N$ with C^2-smooth boundary. For any harmonic function f on D,*

$$\limsup_{\varepsilon \to 0^+} \int_{bD_\varepsilon} |f|^p \, dS = \limsup_{r \to 1^-} (1 - r) \int_D |f(x)|^p \delta^{-r}(x) \, dV. \tag{2.2}$$

Furthermore, when the above limits are finite, then $f \in h^p(D)$ and

$$\int_{bD} |f|^p \, dS = \lim_{r \to 1^-} (1 - r) \int_D |f(x)|^p \delta^{-r}(x) \, dV. \tag{2.3}$$

Proof. If the limit on the left hand side of (2.2) is finite, then $f \in h^p(D)$. Hence

$$\lim_{\varepsilon \to 0^+} \int_{bD_\varepsilon} |f|^p \, dS = \int_{bD} |f|^p \, dS.$$

Write

$$\lambda(\varepsilon) = \int_{bD_\varepsilon} |f|^p \, dS.$$

Then $\lambda(\varepsilon)$ is continuous on $[0, a]$ for any sufficiently small $a > 0$. Therefore,

$$\lim_{r \to 1^-} (1 - r) \int_D |f|^p \delta^{-r} \, dV = \lim_{r \to 1^-} (1 - r) \int_0^a \varepsilon^{-r} \lambda(\varepsilon) \, d\varepsilon = \lambda(0) = \int_{bD} |f|^p \, dS.$$

Now suppose the limit on the right hand side of (2.2) is finite. For any sufficiently small $0 < \varepsilon_1 < \varepsilon_2$, we assume that $\lambda(\varepsilon)$ takes its minimum on $[\varepsilon_1, \varepsilon_2]$ at ε_0. Then

$$(1 - r) \int_D |f|^p \delta^{-r} \, dV \ge (1 - r) \int_{\varepsilon_1 \le \delta \le \varepsilon_2} |f|^p \delta^{-r} \, dV \ge (\varepsilon_2^{1-r} - \varepsilon_1^{1-r}) \lambda(\varepsilon_0).$$

Taking $\liminf_{\varepsilon_1 \to 0^+}$ and then $\limsup_{r \to 1^-}$, we then have

$$\infty > \limsup_{r \to 1^-} (1 - r) \int_D |f(x)|^p \delta^{-r}(x) \, dV \ge \liminf_{\varepsilon_1 \to 0^+} \lambda(\varepsilon_0).$$

It follows that there exists a sequence $\varepsilon_j \to 0^+$ such that $\lambda(\varepsilon_j)$ is bounded. Let $\pi_\varepsilon \colon bD_\varepsilon \to bD$ be the projection along the outward normal direction. Then $f_j(x) = f(\pi_{\varepsilon_j}^{-1}(x))$ is a bounded sequence in $L^p(bD)$. By Alaoglu's theorem, it has a subsequence that converges to some $\tilde{f} \in L^p(bD)$ in the weak* topology. It follows that

$$f(x) = \int_{bD} P(x, y) \tilde{f}(y) \, dS(y).$$

Hence $f \in h^p(D)$ and we can refer back to the first part of the proof. □

We now review the rudiments on the Bergman and Szegö kernels. Let Ω be a bounded domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ be the Bergman space, the space of square integrable holomorphic functions on Ω. The Bergman kernel $K_\Omega(z,w)$ is the reproducing kernel of $A^2(\Omega)$:

$$f(z) = \int_\Omega K_\Omega(z,w) f(w)\, dV, \qquad \forall f \in A^2(\Omega),\ \forall z \in \Omega.$$

Assume that $b\Omega$ is of class C^2. The Hardy space $H^2(\Omega)$ is the space of holomorphic functions on Ω that are also in $h^2(\Omega)$. The Szegö kernel is the reproducing kernel of $H^2(\Omega)$:

$$f(z) = \int_{b\Omega} S_\Omega(z,w) f(w)\, dS, \qquad \forall f \in H^2(\Omega),\ \forall z \in \Omega.$$

It follows from these reproducing properties that

$$K_\Omega(z,z) = \sup\{|f(z)|^2;\ f \in A^2(\Omega), \|f\|_\Omega \le 1\} \tag{2.4}$$

and

$$S_\Omega(z,z) = \sup\{|f(z)|^2;\ f \in H^2(\Omega), \|f\|_{b\Omega} \le 1\}. \tag{2.5}$$

From (2.4), we know that the Bergman kernel has the decreasing property: if $\Omega_1 \subset \Omega_2$, then $K_{\Omega_1}(z,z) \ge K_{\Omega_2}(z,z)$. Combining Lemma 2.1 with (2.5), we have:

Lemma 2.3. *Let $\Omega_1 \subset \Omega_2$ be bounded domains in $\mathbb{C}^n$ with C^2-smooth boundaries. Then there exists a constant $C > 0$ such that*

$$S_{\Omega_2}(z,z) \le C S_{\Omega_1}(z,z) \tag{2.6}$$

for all $z \in \Omega_1$.

3. Weighted L^2-estimates for the $\bar\partial$-operator

In this section, we review relevant weighted L^2-estimates for the $\overline{\partial}$-operator of Hörmander, Demailly, and Berndtsson. We will only state their results for $(0,1)$-forms, which are what we will need later in this paper. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let ψ be a plurisubharmonic function on Ω. Let $L^2(\Omega, e^{-\psi})$ be the Hilbert space of all measurable functions satisfying

$$\|f\|_\psi^2 = \int_\Omega |f|^2 e^{-\psi}\, dV < \infty$$

and let $L^2_{(0,1)}(\Omega, e^{-\psi})$ be the space of $(0,1)$-forms with coefficients in $L^2(\Omega, e^{-\psi})$. Suppose $\partial\overline{\partial}\psi \ge c\partial\overline{\partial}|z|^2$ as currents where c is a positive continuous function. (Here and in what follows, we will drop the letter i from the real $(1,1)$-form $i\partial\overline{\partial}\psi$.) Hörmander's theorem says that for any $\overline{\partial}$-closed $(0,1)$-form f, one can solve the equation

$$\bar\partial u = f \tag{3.1}$$

in the sense of distribution, together with the estimate

$$\int_\Omega |u|^2 e^{-\psi}\, dV \le 2\int_\Omega |f|^2 e^{-\psi}/c\, dV, \tag{3.2}$$

provided the right hand side is finite ([26, Theorem 2.2.1$'$]; see also [27, Lemma4.4.1]). Suppose $\psi \in C^2(\Omega)$. For any $(0,1)$-form f, let

$$|f|_{\partial\overline{\partial}\psi} = \sup\{|\langle f, X\rangle|;\quad X \in T^{0,1}(\Omega),\ |X|_{\partial\overline{\partial}\psi} \le 1\}$$

be the norm induced by the $(1,1)$-form $\partial\overline{\partial}\psi$, where $\langle\cdot,\cdot\rangle$ denotes the pairing of a form and a vector and $|X|_{\partial\overline{\partial}\psi} = \partial\overline{\partial}\psi(\overline{X}, X)$ is the length of X with respect to $\partial\overline{\partial}\psi$. According to

Demailly's reformulation of Hörmander's theorem, one can solve $\overline{\partial} u = f$ with the following estimate

$$\int_\Omega |u|^2 e^{-\psi}\, dV \le \int_\Omega |f|^2_{\partial\bar\partial\psi} e^{-\psi}\, dV, \tag{3.3}$$

provided the right hand side is finite (see [14, Theorem 4.1]). It follows that for any function u in the orthogonal complement of the nullspace $\mathcal{N}(\overline{\partial})$ in $L^2(\Omega, e^{-\psi})$, we have

$$\int_\Omega |u|^2 e^{-\psi}\, dV \le \int_\Omega |\overline{\partial} u|^2_{\partial\bar\partial\psi} e^{-\psi}\, dV. \tag{3.4}$$

The following theorem is a slight reformulation of a result due to Berndtsson ([2, Theorem 2.8]). Berndtsson's proof uses an integration by parts formula related to the $\partial\overline{\partial}$-Bochner-Kodaira technique of Siu (see Section 3 in [41]). We provide a proof here as a simple application of (3.4). Similar approach was used in [3] to prove an estimate of Donnelly-Fefferman [19].

Theorem 3.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Let $\rho \in C^2(\Omega)$ with $\rho < 0$. Suppose that there exists a plurisubhamornic function $\psi \in C^2(\Omega)$ such that*

$$\Theta := (-\rho)\partial\bar\partial\psi + \partial\bar\partial\rho$$

is positive. Let u be the solution to (3.1) that is orthogonal to $\mathcal{N}(\overline{\partial})$ in $L^2(\Omega, e^{-\psi})$. Then for any $0 < r < 1$,

$$(1-r)\int_\Omega |u|^2(-\rho)^{-r} e^{-\psi}\, dV \le \frac{1}{r}\int_\Omega |f|^2_\Theta (-\rho)^{1-r} e^{-\psi}\, dV. \tag{3.5}$$

Proof. Let $\phi = -r\log(-\rho)$ and $\varphi = \phi + \psi$. Then $ue^\phi \perp \mathcal{N}(\overline{\partial})$ in $L^2(\Omega, e^{-\varphi})$. Applying (3.4) to ue^ϕ with weight $e^{-\varphi}$, we have

$$\int_\Omega |u|^2 e^{\phi-\psi}\, dV \le \int_\Omega |\overline{\partial} u + u\overline{\partial}\phi|^2_{\partial\overline{\partial}\varphi} e^{\phi-\psi}\, dV.$$

It remains to show that

$$|\overline{\partial} u + u\overline{\partial}\phi|^2_{\partial\overline{\partial}\varphi} \le r|u|^2 + \frac{1}{r}|\overline{\partial} u|^2_\Theta(-\rho). \tag{3.6}$$

Notice that

$$\begin{aligned}\partial\overline{\partial}\varphi &= \frac{r}{-\rho}\Theta + \frac{1}{r}\partial\phi\wedge\overline{\partial}\phi + (1-r)\partial\overline{\partial}\psi \\ &\ge \frac{r}{-\rho}\Theta + \frac{1}{r}\partial\phi\wedge\overline{\partial}\phi =: \widetilde{\Theta}.\end{aligned}$$

Thus

$$|\overline{\partial} u + u\overline{\partial}\phi|^2_{\partial\overline{\partial}\varphi} \le |\overline{\partial} u + u\overline{\partial}\phi|^2_{\widetilde{\Theta}} = \sup\{\frac{|\langle \overline{\partial} u + u\overline{\partial}\phi, X\rangle|^2}{\frac{r}{-\rho}|X|^2_\Theta + \frac{1}{r}|\langle\overline{\partial}\phi, X\rangle|^2};\quad X \in T^{0,1}(\Omega)\}. \tag{3.7}$$

Inequality (3.6) then follows from (3.7) and the inequalities $|\langle\overline{\partial} u, X\rangle| \le |\overline{\partial} u|_\Theta |X|_\Theta$ and

$$2|u\langle\overline{\partial}\phi, X\rangle\overline{\langle\overline{\partial} u, X\rangle}| \le 2|u||X|_\Theta|\overline{\partial} u|_\Theta|\langle\overline{\partial}\phi, X\rangle| \le \frac{r^2}{-\rho}|u|^2|X|^2_\Theta + \frac{-\rho}{r^2}|\overline{\partial} u|^2_\Theta|\langle\overline{\partial}\phi, X\rangle|^2.$$

$\square$

4. Upper bound estimates

We prove the first part of Theorem 1.1 in this section. For a bounded domain Ω in $\mathbb{C}^n$, the *pluricomplex Green function* with a pole at $w \in \Omega$ is defined by

$$g_\Omega(z,w) = \sup\left\{u(z);\ u \in PSH(\Omega),\, u<0,\, \limsup_{z\to w}(u(z) - \log|z-w|) < \infty\right\}.$$

It is known that for any bounded hyperconvex domain Ω, the pluricomplex Green function $g_\Omega(\cdot, w)\colon \Omega \to [-\infty, 0)$ is a continuous plurisubharmonic function such that $\lim_{z\to b\Omega} g_\Omega(z,w) = 0$ ([15]; see also Chapter 5 in [28]).

Recall that a constant $a \in (0, 1]$ is said to be a *Diederich-Fornæss exponent* for a bounded pseudoconvex domain Ω if there exist a negative plurisubharmonic function φ on Ω and positive constants C_1 and C_2 such that

$$C_1\delta^a(z) \le -\varphi(z) \le C_2\delta^a(z). \tag{4.1}$$

The supremum of all Diederich-Fornæss exponents is called the *Diederich-Fornæss index* of Ω. It follows from the work of Diederich and Fornæss that any bounded pseudoconvex domain with C^2-smooth boundary has a positive Diederich-Fornæss index, which can be arbitrarily small ([16, 17]). It was proved by Demailly [15] that bounded pseudoconvex Lipschitz domains are hyperconvex. More recently, Harrington [23] showed that bounded pseudoconvex Lipchitz domains have indeed positive Diederich-Fornæss indices.

We will make essential use of the following quantitative estimate for the pluricomplex Green function due to Blocki ([4, Theorem 5.2]; see also [25] for prior related results). We provide a proof below, following mostly Blocki's arguments[1], because we will need to refer back to it.

Theorem 4.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose there exists a negative plurisubharmonic function φ on Ω such that*

$$C_1\delta^a(z) \le -\varphi(z) \le C_2\delta^b(z), \qquad z \in \Omega \tag{4.2}$$

for some positive constants C_1, C_2, and $a \ge b$. Then there exists positive constants δ_0 and C such that

$$\{z \in \Omega;\ g_\Omega(z,w) \le -1\} \subset \left\{C^{-1}\delta^{\frac{a}{b}}(w)|\log\delta(w)|^{-\frac{1}{b}} \le \delta(z) \le C\delta^{\frac{b}{a}}(w)|\log\delta(w)|^{\frac{n}{a}}\right\}, \tag{4.3}$$

for any $w \in \Omega$ with $\delta(w) \le \delta_0$.

Proof. Assume that Ω has diameter R. Let $w \in \Omega$ with $r = \delta(w) \le e^{-2}$. Let $z \in \Omega$. Suppose that $\delta = \delta(z) \le r/2$. It follows from comparison with the pluricomplex Green function of $B(w,R)$ that

$$g_\Omega(\zeta,w) \ge \log(|\zeta - w|/R) \tag{4.4}$$

for all $\zeta \in \Omega$. By the maximal property of the pluricomplex Green function, we have

$$g_\Omega(\zeta,w) \ge \frac{\log(2R/r)}{\inf\{|\varphi(\zeta)|;\ \zeta \in \overline{B(w,\frac{r}{2})}\}}\varphi(\zeta) \tag{4.5}$$

on $\Omega \setminus B(w, r/2)$ because the same inequality holds on the boundary. By (4.2),

$$\inf\{|\varphi(\zeta)|;\ \zeta \in \overline{B(w,\frac{r}{2})}\} \ge C(r/2)^a. \tag{4.6}$$

[1]There are slight inaccuracies in the proof of Theorem 5.2 in [4]: The inequality (5.6) and the choice of ε there seem to be incorrect.

Therefore,

$$g_\Omega(z,w) \geq -C\frac{\delta^b}{r^a}\log\frac{1}{r}. \tag{4.7}$$

It follows that

$$\begin{aligned}\{z\in\Omega;\ g_\Omega(z,w)\leq -1\} &\subset \left\{\delta(z) > \frac{r}{2} \text{ or } \delta(z) \geq C r^{\frac{a}{b}} (\log(1/r))^{-\frac{1}{b}}\right\} \\ &\subset \left\{\delta(z) \geq C^{-1}\delta^{\frac{a}{b}}(w)|\log\delta(w)|^{-\frac{1}{b}}\right\}, \end{aligned}\tag{4.8}$$

provided the last constant C is sufficiently large.

Now suppose that $e^{-2}\geq \delta(z)\geq 2r$. It follows from (4.7) that for any $0<\varepsilon<r/2$,

$$\inf_{\delta(\zeta)=\varepsilon} g_\Omega(\zeta,w) \geq -C\frac{\varepsilon^b}{r^a}\log\frac{1}{r}. \tag{4.9}$$

We also obtain from (4.7) that

$$g_\Omega(w,z) \geq -C\frac{r^b}{\delta^a}\log\frac{1}{\delta}, \tag{4.10}$$

by reversing the rôles of z and w. By Theorem 5.1 in [4], we have

$$g_\Omega(z,w) \geq -C\frac{\log\frac{1}{\varepsilon}}{\log\frac{r}{2\varepsilon}}\left(\frac{\varepsilon^b}{r^a}\log\frac{1}{r} + \frac{r^{\frac{b}{n}}}{\delta^{\frac{a}{n}}}(\log\frac{1}{\varepsilon})^{1-\frac{1}{n}}(\log\frac{1}{\delta})^{\frac{1}{n}}\right). \tag{4.11}$$

We have followed closely Blocki's proof thus far. Here is where we start to deviate. Suppose further that

$$\delta(z) \geq r^{\frac{b}{a}}\left(\log\frac{1}{r}\right)^{\frac{n}{a}}. \tag{4.12}$$

Set

$$\varepsilon = \frac{1}{2}\frac{r^{\frac{1}{n}+\frac{a}{b}}}{\delta^{\frac{a}{bn}}}\left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\frac{1}{b}}. \tag{4.13}$$

Since $\delta\geq 2r$, $\log\log\frac{1}{r}\leq\log\frac{1}{r}$, and

$$\frac{1}{n}+\frac{a}{b}-\frac{a}{bn}\geq 1, \tag{4.14}$$

we have

$$\varepsilon \leq \frac{r^{\frac{1}{n}+\frac{a}{b}-\frac{a}{bn}}}{2^{1+\frac{a}{bn}}} < \frac{1}{2}r$$

as required. From (4.12) and (4.13), we know that

$$\frac{r}{2\varepsilon} \geq r^{1-\frac{a}{b}}\left(\log\frac{1}{r}\right)^{\frac{1}{b}}\left(\frac{\log\frac{1}{r}}{\log\log\frac{1}{r}}\right)^{\frac{1}{b}} \geq C\left(\log\frac{1}{r}\right)^{\frac{1}{b}},$$

provided $r=\delta(w)$ is sufficiently small. (Recall that $b\leq a$.) Therefore,

$$\log\frac{r}{2\varepsilon} \geq C\log\log\frac{1}{r}. \tag{4.15}$$

It is easy to see that

$$\log\frac{1}{\varepsilon}\leq C\log\frac{1}{r} \quad\text{and}\quad \log\frac{1}{\delta}\leq C\log\frac{1}{r}. \tag{4.16}$$

Combining (4.13), (4.15), and (4.16) with (4.11), we then obtain

$$g_\Omega(z,w) \ge -C\frac{r^{\frac{b}{n}}}{\delta^{\frac{a}{n}}}\log\frac{1}{r}.$$

Therefore,

$$\{g_\Omega(z,w) < -1\} \subset \{z \in \Omega;\ \delta(z) \ge e^{-2} \text{ or } \delta(z) \le C\delta^{b/a}(w)|\log\delta(w)|^{n/a}\}.$$

Together with (4.8), we then obtain (4.3) by choosing a sufficiently small δ_0 and a sufficiently large C. □

We also need the following localization of the Bergman kernel ([12, Lemma 4.2]; also [24, Proposition 3.6]).

Proposition 4.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then there exists a positive constant C such that for any $w \in \Omega$,*

$$K_\Omega(w,w) \ge CK_{\{g_\Omega(\cdot,w)<-1\}}(w,w).$$

To illustrate the idea of the proof, we first prove the following weaker version of Theorem 1.1 (1):

Proposition 4.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-smooth boundary. Suppose that the Diederich-Fornæss index of Ω is β. Then for any $a \in (0,\ \beta)$, there exists a constant $C > 0$ such that*

$$\frac{S(z,z)}{K(z,z)} \le C\delta(z)|\log(\delta(z))|^{n/a}. \tag{4.17}$$

Proof. By the definition of the Diederich-Fornæss index, there exists a negative plurisubharmonic function φ satisfying (4.1). By Theorem 4.1, there exists a positive constant C such that

$$\{g_\Omega(\cdot,w) < -1\} \subset \{\delta(\cdot) < C\delta(w)|\log\delta(w)|^{n/a}\} \tag{4.18}$$

for any $w \in \Omega$ sufficiently closed to the boundary. Therefore, for any $f \in H^2(\Omega)$,

$$\int_{\{g_\Omega(\cdot,w)<-1\}} |f|^2\, dV \le \int_0^{C\delta(w)|\log\delta(w)|^{n/a}} d\varepsilon \int_{b\Omega_\varepsilon} |f|^2\, dS \le C\|f\|^2_{b\Omega}\delta(w)|\log\delta(w)|^{n/a}.$$

It then follows from the extremal properties (2.4) and (2.5) that

$$S(w,w) \le C\delta(w)|\log\delta(w)|^{n/a}K_{\{g(\cdot,w)<-1\}}(w,w).$$

Applying Proposition 4.2, we then conclude the proof of the proposition. □

To get from Proposition 4.3 to the first statement of Theorem 1.1, we use Lemma 2.3 to localize the Szegö kernel and then apply the following fact: For any $z_0 \in b\Omega$ and $a \in (0,\ 1)$, there exist a defining function r of Ω and a neighborhood U of z_0 such that $\varphi_2 = -(-r)^a$ is strictly plurisubharmonic on $U \cap \Omega$ ([16], Remark on p. 133). The problem is that this function φ_2 is not an exhaustion function of $U \cap \Omega$ and thus one cannot directly apply Theorem 4.1. We now show how to overcome this difficulty and prove Theorem 1.1 (1).

Let $\chi(t)$ be a smooth function such that $\chi(t) = 0$ when $t \le 1$, $\chi(t) > 0$ is strictly increasing and convex when $t > 1$. We may further assume that $\chi(t) = \exp(-1/(t-1))$ when $t \in (1,\ 5/4)$ so that $(\chi(t))^b \in C^\infty(\mathbb{R})$ for any positive number b. Let

$$\varphi_1(z) = -(-r)^a + M\chi(|z-z_0|^2/m^2).$$

Then φ_1 is strictly plurisubharmonic on $U \cap \Omega$. Write $g(z) = M\chi(|z - z_0|^2/m^2)$. Let

$$\widetilde{\Omega} = \{z \in \Omega; \quad \varphi_1(z) = -(-r)^a + g < 0\}.$$

By choosing m sufficiently small and M sufficiently large, we know that $B(z_0, m) \cap \Omega \subset \widetilde{\Omega}$ and $\widetilde{\Omega} \subset B(z_0, 2m)$. Furthermore, $\widetilde{\Omega}$ is pseudoconvex with a C^2-smooth defining function

$$\tilde{r} = r + g^{1/a}$$

(see, for example, [1, pp. 470–471]).

Evidently, φ_1 is a plurisubharmonic exhaustion function for $\widetilde{\Omega}$. However, φ_1 does not satisfy (4.1). In fact, it is easy to show that there exists a positive constant C_1 such that

$$C_1|\tilde{r}| \leq -\varphi_1 \leq |\tilde{r}|^a, \quad z \in \widetilde{\Omega}. \tag{4.19}$$

If we directly invoke Theorem 4.1 with (4.19), we obtain

$$\{g_{\widetilde{\Omega}}(\cdot, w) < -1\} \subset \{\delta(\cdot) < C\delta^a(w)|\log \delta(w)|^n\}.$$

Consequently, we have as in the proof of Proposition 4.3 that

$$S(z,z)/K(z,z) \leq C\delta^a(z)|\log \delta(z)|^n,$$

which is even weaker than (4.17).

Instead of directly appealing to Theorem 4.1, we proceed as follows. We follow the proof of Theorem 4.1 with Ω replaced by $\widetilde{\Omega}$, and with $\delta(z) = \delta_{\widetilde{\Omega}}(z)$ now denoting the Euclidean distance to $b\widetilde{\Omega}$. Notice that $C^{-1}\delta \leq |\tilde{r}| \leq C\delta$ on $\widetilde{\Omega}$. Assume that $|w - z_0| < m$. Applying (4.5) to the function φ_1, we have

$$\inf_{\delta(\zeta)=\varepsilon} g_{\widetilde{\Omega}}(\zeta, w) \geq \frac{\log(2R/r)}{\inf\{|\varphi_1(\zeta)|;\ \zeta \in \overline{B(w, \frac{r}{2})}\}} \inf_{\delta(\zeta)=\varepsilon} \varphi_1(\zeta) \geq -C\frac{\varepsilon^a}{r^a}\log\frac{1}{r}, \tag{4.20}$$

which is our analogue in this case to (4.9). (Here we have $a = b$.) Now applying (4.5) to the function $\varphi_2 = -(-r)^a$ with the rôle of z and w reversed, we have

$$g_{\widetilde{\Omega}}(\zeta, z) \geq \frac{\log(2R/\delta)}{\inf\{|\varphi_2(\zeta)|;\ \zeta \in \overline{B(z, \frac{\delta}{2})}\}}\varphi_2(\zeta).$$

on $\widetilde{\Omega} \setminus B(z, \delta/2)$. It follows that

$$g_{\widetilde{\Omega}}(w, z) \geq -C\frac{r^a}{\delta^a}\log\frac{1}{\delta}, \tag{4.21}$$

which plays the rôle of (4.10) in this case. Following exactly the same lines for the rest of the proof of Theorem 4.1, we then obtain

$$\{z \in \widetilde{\Omega};\ g_{\widetilde{\Omega}}(z, w) < -1\} \subset \{z \in \widetilde{\Omega};\ \delta(z) \leq C\delta(w)|\log \delta(w)|^{n/a}\}.$$

From the proof of Proposition 4.3, we then have

$$\frac{S_{\widetilde{\Omega}}(w,w)}{K_{\widetilde{\Omega}}(w,w)} \leq C\delta(w)|\log(\delta(w))|^{n/a},$$

when w is sufficiently close to z_0. By the localization property of the Bergman kernel (see the proof of Theorem 1 in [35]; also [18, Proposition 1]), $K_\Omega(w,w) \geq CK_{\widetilde{\Omega}}(w,w)$. Together with Lemma 2.3, we then conclude the proof of the first statement in Theorem 1.1.

5. Lower bound estimates

Recall that a continuous function ρ is said to be a defining function of a domain $\Omega \subset \mathbb{C}^n$ if $\Omega = \{z \in \mathbb{C}^n;\ \rho(z) < 0\}$ and $C^{-1}\delta \le \rho \le C\delta$ for a constant $C > 0$. We also assume the defining function ρ to be in the same smoothness class as that of the boundary $b\Omega$. A bounded domain $\Omega \subset \mathbb{C}^n$ is *δ-regular* if there exist a neighborhood U of $b\Omega$, a bounded continuous plurisubharmonic function φ on $U \cap \Omega$, and a defining function ρ of Ω such that

$$\partial\overline{\partial}\varphi \ge \rho^{-1}\partial\overline{\partial}\rho \tag{5.1}$$

on $U \cap \Omega$ as currents. By adding $|z|^2$ to φ, we may assume that it is strictly plurisubharmonic. By Richberg's approximation theorem ([39, Satz 4.3]), we may further assume that $\varphi \in C^\infty(\Omega)$.

Proposition 5.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a δ-regular domain. Then Ω is hyperconvex with a positive Diederich-Fornæss index.*

Proof. Let φ and ρ be the functions that satisfy (5.1). Assume $0 \le \varphi \le M$ for some positive constant M. Let $\psi = e^\varphi$ and $K = e^M$. Then

$$1 \le \psi \le K, \quad \partial\psi \wedge \overline{\partial}\psi \le K\partial\overline{\partial}\psi, \quad \text{and} \quad \partial\overline{\partial}\psi \ge \frac{\partial\overline{\partial}\rho}{\rho} + K^{-1}\partial\psi \wedge \overline{\partial}\psi, \tag{5.2}$$

on $U \cap \Omega$. Let

$$\tilde{\rho} = \rho e^{-\psi} \quad \text{and} \quad r = -(-\tilde{\rho})^\eta.$$

It follows from a simple (formal) computation and (5.2) that

$$\begin{aligned}
\partial\overline{\partial}r =& \eta(-\tilde{\rho})^\eta \left(\partial\overline{\partial} - \log(-\tilde{\rho}) - \eta\frac{\partial\tilde{\rho} \wedge \overline{\partial}\tilde{\rho}}{\tilde{\rho}^2}\right) \\
&\ge \eta(-\tilde{\rho})^\eta \left(\frac{1}{K}\partial\psi \wedge \overline{\partial}\psi + \frac{\partial\rho \wedge \overline{\partial}\rho}{\rho^2} - \eta\frac{\partial\tilde{\rho} \wedge \overline{\partial}\tilde{\rho}}{\tilde{\rho}^2}\right) \\
&\ge \eta(-\tilde{\rho})^\eta \left((1-\eta)\frac{\partial\rho \wedge \overline{\partial}\rho}{\rho^2} + \eta\frac{\partial\rho}{\rho} \wedge \overline{\partial}\psi + \eta\partial\psi \wedge \frac{\overline{\partial}\rho}{\rho} + (\frac{1}{K} - \eta)\partial\psi \wedge \overline{\partial}\psi\right).
\end{aligned}$$

We then obtain from the Schwarz inequality that $\partial\overline{\partial}r$ is a positive current on $U \cap \Omega$, provided η is sufficiently small. The extension of r to the whole domain Ω is standard (see [16, p. 133]). $\square$

Proposition 5.2. *Let Ω be a smooth pseudoconvex bounded domain in $\mathbb{C}^n$. If Ω has a defining function that is plurisubharmonic on $b\Omega$ or if $b\Omega$ is of D'Angelo finite type, then Ω is δ-regular.*

Proof. If Ω has a defining function ρ that is plurisubharmonic on $b\Omega$. Then $\partial\overline{\partial}\rho \ge C\rho\partial\overline{\partial}|z|^2$. Therefore in this case, we can choose $\varphi(z) = C|z|^2$ with a sufficiently large C.

The case when Ω is of D'Angelo finite type is a consequence of Catlin's construction of bounded plurisubharmonic function [10] (see [44, p. 464] for a related discussion): There exist positive constants $\tau < 1$, $C > 0$, and a smooth bounded plurisubharmonic function λ on Ω such that

$$\partial\bar{\partial}\lambda \ge C\frac{\partial\bar{\partial}|z|^2}{|\rho|^\tau}. \tag{5.3}$$

By Oka's lemma, we can choose a defining function ρ such that $\partial\overline{\partial}(-\log(-\rho)) \ge \partial\overline{\partial}|z|^2$. It then follows from a theorem of Diederich-Fornaess that for any sufficiently small η,

$$\partial\bar{\partial}(-(-\rho)^\eta) \ge C\eta|\rho|^\eta \left(\partial\overline{\partial}|z|^2 + |\rho|^{-2}\partial\rho \wedge \overline{\partial}\rho\right) \tag{5.4}$$

for some positive constant C ([16], Theorem 1 and its proof; compare also [8, Lemma 2.2]).

Now we fix an $\eta \in (0,\ \tau)$. Write $N = |\partial\rho|^{-1} \sum \rho_{\bar{z}_j}\partial/\partial z_j$. For any $(1,0)$-vector X, write $X_N = \langle X, N\rangle N$ and $X_T = X - X_N$. By the pseudoconvexity of $b\Omega$, we have

$$\rho^{-1}\partial\overline{\partial}\rho(X, \overline{X}) \le C(|X|^2 + |\rho|^{-1}|X||X_N|) \le C(|\rho|^{-\eta}|X|^2 + |\rho|^{-2+\eta}|X_N|^2)$$

The desirable function is then given by

$$\varphi = C(\lambda - (-\rho)^\eta)$$

for any sufficiently large $C > 0$. □

We are now in position to prove the second part of Theorem 1.1. Let κ be a standard Friedrichs mollifier. Let ε_j be a decreasing sequence of positive number tending to 0. Let w be a point in Ω, sufficiently closed to the boundary $b\Omega$. Let $g_j = g_\Omega(\cdot, w) * \kappa_{\varepsilon_j}$. Then g_j is a decreasing sequence of plurisubharmonic functions on $\Omega_j = \{z \in \Omega;\ \delta(z) < \varepsilon_j\}$ with limit $g_\Omega(\cdot, w)$. By Oka's lemma, Ω_j is pseudoconvex. Let

$$\psi = 2ng_\Omega(\cdot, w) - \log(-g_\Omega(\cdot, w) + 1) + \varphi \quad \text{and} \quad \psi_j = 2ng_j - \log(-g_j + 1) + \varphi,$$

where φ is the smooth bounded strictly plurisubharmonic function, obtained from the δ-regularity assumption, such that $\partial\overline{\partial}\varphi \ge \rho^{-1}\partial\overline{\partial}\rho$ for a defining function ρ of Ω. Clearly ψ_j is a plurisubharmonic function on Ω_j. Moreover,

$$\partial\bar{\partial}\psi_j \ge \partial\log(-g_j + 1) \wedge \bar{\partial}\log(-g_j + 1) \tag{5.5}$$

and $\Theta_j = (-\rho)\partial\bar{\partial}\psi_j + \partial\bar{\partial}\rho$ is positive on Ω_j. Let $\chi : \mathbb{R} \to [0, 1]$ be a C^∞ cut-off function such that $\chi|_{(-\infty,-1)} = 1$ and $\chi|_{(0,\infty)} = 0$. Put

$$v_j = \bar{\partial}\chi(-\log(-g_j))\frac{K_\Omega(\cdot, w)}{\sqrt{K_\Omega(w, w)}}.$$

Let u_j be the solution to $\overline{\partial}u_j = v_j$ that is in the orthogonal complement of $\mathcal{N}(\overline{\partial})$ in $L^2(\Omega_j, e^{-\psi_j})$. By Demailly's estimate (3.3),

$$\int_{\Omega_j} |u_j|^2 e^{-\psi_j}\, dV \le \int_{\Omega_j} |v_j|^2_{\partial\bar{\partial}\psi_j} e^{-\psi_j}\, dV.$$

It follows from (5.5) that the right hand side is uniformly bounded from above, independent of j. Passing to a subsequence, we may assume that u_j converges to $u \in L^2(\Omega, e^{-\psi})$ in the weak* topology. (We extend $u_j = 0$ on $\Omega \setminus \Omega_j$.) Let

$$f = \chi(-\log(-g_\Omega(\cdot, w)))K_\Omega(\cdot, w)/K_\Omega(w, w)^{1/2} - u.$$

Then f is holomorphic on Ω. Since u is holomorphic in a neighborhood of w and $g_\Omega(z, w) = \log|z - w| + O(1)$ near w,

$$u(w) = 0.$$

By Theorem 3.1, for any $0 < r < 1$, we have

$$
\begin{aligned}
(1-r)\int_{\Omega_j} |u_j|^2(-\rho)^{-r}e^{-\psi_j}\,dV &\le \frac{1}{r}\int_{\Omega_j} |v_j|^2_{\Theta_j}(-\rho)^{1-r}e^{-\psi_j}\,dV \\
&\le \frac{C}{r}\int_{\operatorname{supp}\bar\partial\chi(\cdot)} \frac{|K_\Omega(\cdot,w)|^2}{K_\Omega(w,w)}(-\rho)^{-r}\,dV. \qquad (5.6)
\end{aligned}
$$

By Theorem 4.1,

$$
\begin{aligned}
\operatorname{supp}\bar\partial\chi(\cdot) &\subset \{-e \le g_j(\cdot,w) \le -1\} \subset \{g_\Omega(\cdot,w) \le -1\} \\
&\subset \{C^{-1}|\rho(w)||\log(-\rho(w))|^{-1/a} \le |\rho|\},
\end{aligned}
$$

where a is a Diederich-Fornæss exponent for Ω. Therefore, passing to the limit, we have

$$
(1-r)\int_{\Omega} |u|^2(-\rho)^{-r}e^{-\psi}\,dV \le \frac{C}{r}\cdot\frac{|\log(-\rho(w))|^{r/a}}{|\rho(w)|^r}. \qquad (5.7)
$$

Notice that f is a holomorphic function on Ω such that $f(w) = K_\Omega(w,w)^{1/2}$ and $f = -u$ near $b\Omega$. Since $e^{-\psi} \ge e^{-\varphi} \ge C > 0$, by Lemma 2.2 (and its proof), $f \in H^2(\Omega)$. Combining with (5.7), we have

$$
\begin{aligned}
\int_{b\Omega} |f|^2\,dS &\le C\lim_{r\to 1^-}(1-r)\int_\Omega |f|^2(-\rho)^{-r}e^{-\psi}\,dV \\
&= C\lim_{r\to 1^-}(1-r)\int_\Omega |u|^2(-\rho)^{-r}e^{-\psi}\,dV \\
&\le C\frac{|\log(-\rho(w))|^{1/a}}{|\rho(w)|}.
\end{aligned}
$$

It then follows from the extremal property (2.5) of the Szegö kernel that

$$
S_\Omega(w,w) \ge C\frac{|\rho(w)|}{|\log(-\rho(w))|^{1/a}}\cdot K_\Omega(w,w).
$$

This concludes the proof of Theorem 1.1.

The proof of Theorem 1.3 is similar that of Theorem 1.1. The only difference is that, instead of Theorem 4.1, we use the following well known estimate for the pluricomplex Green function on convex domains:

$$
\{g_\Omega(\cdot,w) \le -1\} \subset \{C^{-1}\delta(w) \le \delta(\cdot) \le C\delta(w)\}
$$

(see [4], Theorem 5.4).

Remark. It follows from the Ohsawa-Takegoshi extension theorem [37] that for any bounded pseudoconvex Lipschitz domain Ω in $\mathbb{C}^n$, $K(z,z) \ge C\delta^{-2}(z)$ for some constant $C > 0$. Ohsawa [36] conjectured that the analogue estimate $S(z,z) \ge C\delta^{-1}(z)$ holds. Theorem 1.3 confirms this conjecture for convex domains. Moreover, Theorem 1.1 implies that for a bounded δ-regular domain Ω with C^2-smooth boundary,

$$
S(z,z) \ge C\delta^{-1}(z)|\log\delta(z)|^{-1/a}, \qquad (5.8)
$$

where a is any Diederich-Fornæss exponent of Ω.

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